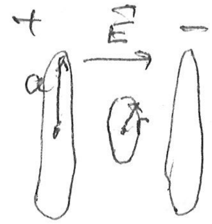
EM A3

(a) $E = \frac{\sigma}{\varepsilon_0} = \frac{\sigma_0 + kt}{\varepsilon_0}$



(b)

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{E}\cdot d\vec{a}$$

$$B\cdot 2\pi r = \mu_0 \varepsilon_0 \frac{d}{dt}\left(\frac{\sigma}{\varepsilon_0}\cdot \pi r^2\right)$$

$$B\cdot 2\pi r = \mu_0 \varepsilon_0 \frac{\cancel{\sigma} k}{\varepsilon_0}\cdot \pi r^2 \qquad \text{Since } \frac{d\sigma}{dt} = k$$

$$B = \frac{\mu_0 k r \pi}{2\pi} = \frac{\mu_0 k r}{2}$$

EM A4

$V=0, \quad \vec{A} = A_0 \hat{x} \sin(kz-\omega t)$

$\lambda = \lambda_0 \cos(kz-\omega t)$

$\vec{\nabla}\lambda = -\hat{z}\, k\lambda_0 \sin(kz-\omega t)$

$\frac{\partial \lambda}{\partial t} = \omega \lambda_0 \sin(kz-\omega t)$

$\vec{A}' = A_0 \hat{x} \sin(kz-\omega t) - \hat{z} k\lambda_0 \sin(kz-\omega t)$

$= (A_0 \hat{x} - k\lambda_0 \hat{z}) \sin(kz-\omega t) \quad (\vec{A}' = \vec{A} + \vec{\nabla}\lambda)$

$V' = V - \frac{\partial \lambda}{\partial t} = -\omega \lambda_0 \sin(kz-\omega t)$

$\vec{\nabla}\cdot\vec{A}' = -k^2 \lambda_0 \cos(kz-\omega t)$ → not the Coulomb gauge

$\frac{\partial V'}{\partial t} = \omega^2 \lambda_0 \cos(kz-\omega t)$

$\vec{\nabla}\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial V'}{\partial t} = -k^2 \lambda_0 \cos(kz-\omega t) + \frac{\omega^2}{c^2}\lambda_0 \cos(kz-\omega t)$

$= 0$ since $k = \frac{\omega}{c}$

→ Lorentz gauge.

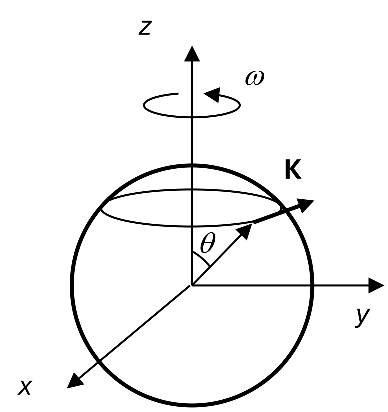
A1. A linear homogeneous dielectric of arbitrary shape is placed in a static electric field. There is no free charge density in the dielectric. Show that the polarization charge can appear only on the surface of this dielectric but not in the bulk. What happens if the dielectric is inhomogeneous, i.e. dielectric permittivity $\varepsilon(\mathbf{r})$ is dependent on coordinates? Write expression for the volume polarization charge density in terms of $\varepsilon(\mathbf{r})$ and electric field $E(\mathbf{r})$.

Solution: A volume polarization-charge density is given by $\rho_P = -\nabla\cdot\mathbf{P}$, where $\mathbf{P}$ is polarization which is related to electric displacement $\mathbf{D}$ as $\mathbf{P} = \mathbf{D} - \varepsilon_0\mathbf{E}$. For a linear dielectric, we have $\mathbf{D} = \varepsilon\mathbf{E}$, where ε is dielectric permittivity. Therefore $\mathbf{P} = \left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\mathbf{D}$ and the volume polarization charge density reads $\rho_P = -\nabla\cdot\left[\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\mathbf{D}\right]$. If the dielectric is homogeneous, i.e. ε is space independent, we have $\rho_P = -\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\nabla\cdot\mathbf{D} = 0$. The latter equality follows from Gauss's law which tells us that in the absence of free charge density $\nabla\cdot\mathbf{D} = 0$. Therefore, the volume polarization charge density vanishes, so that the polarization charge can appear only on the surface.

If the dielectric is inhomogeneous and the dielectric constant $\varepsilon(\mathbf{r})$ depends on coordinates, polarization charge is in general non-zero. It is given by $\rho_P = -\varepsilon\mathbf{E}\cdot\nabla\left(1-\frac{\varepsilon_0}{\varepsilon}\right)$.

A2. Find the magnetic moment of a spherical shell of radius R rotating with frequency ω and having a constant surface charge density σ.

Solution:



Use the coordinates with the z axis along the rotating axis and the origin at the center of the sphere. The surface current density on the spherical shell in the spherical coordinates is

$$\mathbf{K} = R\sigma\omega\sin\theta\,\hat{\boldsymbol{\phi}}.$$

where $\hat{\boldsymbol{\phi}}$ is a polar unit vector. The magnetic dipole moment is obtained by integrating over the sphere so that

$$\mathbf{m} = \frac{1}{2}\int\left[\mathbf{R}\times\mathbf{K}(\mathbf{r})\right]da = \frac{1}{2}\int_0^{\pi} R\sigma\omega\sin\theta\left(\mathbf{R}\times\hat{\boldsymbol{\phi}}\right)2\pi R^2\sin\theta\, d\theta =$$

$$= \hat{\mathbf{z}}\pi R^4\sigma\omega\int_0^{\pi}\sin^3\theta\, d\theta = \frac{4\pi}{3}R^4\sigma\omega\hat{\mathbf{z}}.$$

B1. A slab of uniform volume charge density ρ_0 has its bottom and top surfaces at $z = -d$ and $z = 0$, respectively. Another slab of uniform volume charge density $-\rho_0$ is placed above, so that its bottom and top surfaces at $z = 0$ and $z = d$, respectively. Find the electric field and electrostatic potential in all space. Sketch the electric field and the electrostatic potential as a function of z.

Solution:

The electric field depends only on z, and therefore Gauss's law takes form

$$\frac{\partial E}{\partial z} = \frac{\rho(z)}{\varepsilon_0},$$

where $\rho(z) = \rho_0$ for $-d < z < 0$, $\rho(z) = -\rho_0$ for $0 < z < d$, and $\rho(z) = 0$ for $z > |d|$. We integrate this equation from $-d$ to z ($0 < z < d$), taking into account the fact that, due to the charge neutrality of the system, there is no electric field at $z > |d|$. We therefore have

$$E(z) = \frac{1}{\varepsilon_0}\begin{cases}(z+d), & -d < z < 0,\\ (d-z), & 0 < z < d.\end{cases}$$

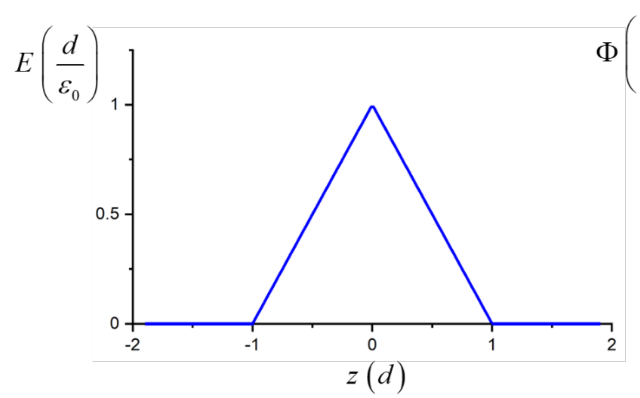
Assuming $\Phi(z) = 0$ at $z \le -d$ as a reference, we integrate $E(z)$ from $-d$ to z ($-d < z < 0$), and obtain:

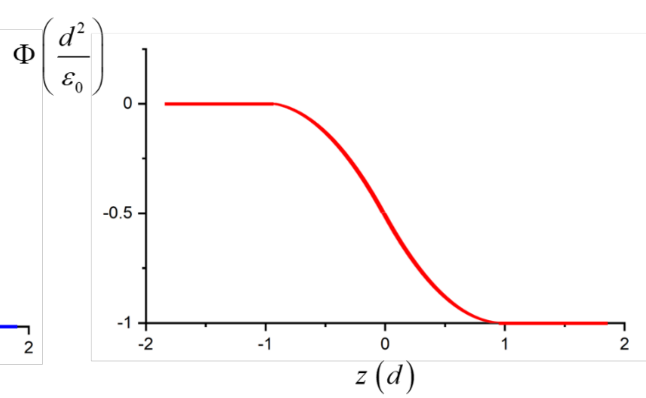
$$\Phi(z) = -\int_{-d}^{z} E(z)dz = -\frac{1}{\varepsilon_0}\int_{-d}^{z}(z+d)dz = -\frac{1}{2\varepsilon_0}(z+d)^2 \text{ at } -d < z < 0.$$

Now, using $\Phi(0) = -\frac{1}{2\varepsilon_0}d^2$, we integrate $E(z)$ from 0 to z ($0 < z < d$) and obtain:

$$\Phi(z) = -\frac{1}{2\varepsilon_0}d^2 - \frac{1}{\varepsilon_0}\int_0^z (d-z)dz = -\frac{1}{\varepsilon_0}d^2 + \frac{1}{2\varepsilon_0}(d-z)^2 \text{ at } 0 < z < d.$$

The field and potential are sketched below:





B2. Perfect conductors are known to completely screen magnetic fields in their interior by creating persistent surface currents. Consider a semi-infinite perfect conductor occupying volume $z < 0$ and having a flat surface at $z = 0$. An infinitely long straight wire is suspended at $z = d$ above the conductor surface and carries current I parallel to the x direction. Find the surface current $\mathbf{K}$ induced on the perfect conductor surface. Show that the total current on the surface is equal in magnitude to the current in the wire but flowing in the opposite direction.

Solution:

The surface current can be found from the boundary conditions which require a parallel component of magnetic field $\mathbf{H}$ to have discontinuity on the surface carrying surface current such that

$$\mathbf{H}_2^{\parallel} - \mathbf{H}_1^{\parallel} = (\mathbf{K} \times \mathbf{n}) ,$$

where $\mathbf{n}$ is the normal to the surface pointing from region 1 to region 2. Multiplying this equation by $\mathbf{n}$ from left, we find:

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K} .$$

This follows from the fact that $\mathbf{n} \times \mathbf{H}_{1,2}^{\parallel} = \mathbf{n} \times \mathbf{H}_{1,2}$ and $\mathbf{n} \times (\mathbf{K} \times \mathbf{n}) = \mathbf{K}(\mathbf{n} \cdot \mathbf{n}) - \mathbf{n}(\mathbf{n} \cdot \mathbf{K}) = \mathbf{K}$.

In our case, assuming that $\mathbf{n} = \hat{\mathbf{z}}$ is the outward normal to the surface of the perfect conductor, $\mathbf{H}_1 = 0$ is the field inside the conductor, and $\mathbf{H} = \mathbf{H}_2$ is the field produced by the wire. We therefore have

$$\mathbf{K} = \hat{\mathbf{z}} \times \mathbf{H} .$$

Field H produced by the straight infinite wire is

$$\mathbf{H} = \frac{I}{2\pi s} \hat{\boldsymbol{\phi}} ,$$

where on the surface of the conductor $s = \sqrt{d^2 + y^2}$ and $\hat{\boldsymbol{\phi}} = \dfrac{d\hat{\mathbf{y}} - y\hat{\mathbf{z}}}{\sqrt{d^2 + y^2}}$.

We therefore obtain

$$\mathbf{K} = \frac{I}{2\pi(d^2 + y^2)} \hat{\mathbf{z}} \times (d\hat{\mathbf{y}} - y\hat{\mathbf{z}}) = -\frac{Id}{2\pi(d^2 + y^2)} \hat{\mathbf{x}} .$$

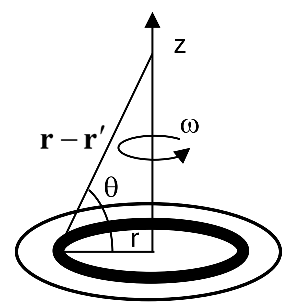
The total current on the surface $\mathbf{I}_s$ is obtained by the integration of current $\mathbf{K}$. We find

$$\mathbf{I}_s = \int_{-\infty}^{\infty} \mathbf{K} dy = -\frac{Id}{2\pi} \hat{\mathbf{x}} \int_{-\infty}^{\infty} \frac{dy}{(d^2 + y^2)} = -I\hat{\mathbf{x}} ,$$

as expected.

B3. A disk of radius R has a uniform surface charge density σ and is rotated with angular frequency ω around the axis z perpendicular to the disk and crossing it at the center. Find the magnetic field **B** on the axis z.

Solution:



The total charge in the shaded ring is $dq = \sigma(2\pi r)dr$. The time of revolution is $dt = 2\pi/\omega$. Therefore, the current in the ring is $I = dq/dt = \sigma\omega r dr$. The magnetic field produced by the ring on the z axis has only non-vanishing z component which is given by

$$d\mathbf{B} = \frac{\mu_0}{4\pi} I \int \frac{dl' \cos\theta}{|\mathbf{r}-\mathbf{r}'|^2}\hat{\mathbf{z}} = \frac{\mu_0}{4\pi} I(2\pi r)\frac{\cos\theta \hat{\mathbf{z}}}{r^2+z^2} =$$

$$= \frac{\mu_0 I}{2}\frac{r^2\hat{\mathbf{z}}}{(r^2+z^2)^{3/2}} = \frac{\mu_0\sigma\omega}{2}\frac{r^3 dr\hat{\mathbf{z}}}{(r^2+z^2)^{3/2}}.$$

Therefore, the total field produced by the disk is

$$\mathbf{B} = \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{r^3 dr}{(r^2+z^2)^{3/2}}\hat{\mathbf{z}}.$$

Let $u \equiv r^2$ so that $du = 2rdr$ and hence

$$\mathbf{B} = \frac{\mu_0\sigma\omega}{4}\int_0^{R^2}\frac{udu}{(u+z^2)^{3/2}} = \frac{\mu_0\sigma\omega}{4}\left[2\frac{u+2z^2}{\sqrt{u+z^2}}\right]_0^{R^2}\hat{\mathbf{z}} = \frac{\mu_0\sigma\omega}{2}\left[\frac{R^2+2z^2}{\sqrt{R^2+z^2}} - 2z\right]\hat{\mathbf{z}}.$$

B4: An electrical circuit consists of a long thin cylindrical conducting shell of radius R and a parallel return wire of radius r on axis inside. The current is assumed distributed uniformly throughout the cross section of the wire. Calculate the self-inductance per unit length. Assume that the permeability of the conductor is μ_0. What is the self-inductance if the inner conductor is a thin hollow tube of the same radius r?

Solution:

The system has axial symmetry and the magnetic field can be easily found from Ampere's law:

$$\mathbf{B}(s) = \frac{\mu_0 I}{2\pi r^2} s\,\hat{\boldsymbol{\phi}}, \quad 0 < s < r;$$
$$\mathbf{B}(s) = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}}, \quad r < s < R;$$
$$\mathbf{B}(s) = 0, \quad s > R.$$

where $\hat{\boldsymbol{\phi}}$ is a polar unit vector and s is the distance from the axis. Now we calculate the magnetic energy per unit width

$$W = \frac{1}{2\mu_0}\int_0^{2\pi}\int_0^{\infty} B^2 s ds d\phi = \frac{1}{2\mu_0} 2\pi\left[\int_0^r \left(\frac{\mu_0 I}{2\pi r^2} s\right)^2 s ds + \int_r^R \left(\frac{\mu_0 I}{2\pi s}\right)^2 s ds\right] = \frac{\mu_0 I^2}{4\pi}\left[\frac{1}{4} + \ln\left(\frac{R}{r}\right)\right].$$

On the other hand, we know that

$$W = \frac{1}{2} L I^2$$

where L is the self-inductance. This leads to

$$L = \frac{\mu_0}{2\pi}\left[\frac{1}{4} + \ln\left(\frac{R}{r}\right)\right].$$

If the inner conductor is a thin hollow tube that the magnetic field inside the tube is zero and hence

$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{R}{r}\right).$$

a) In spherical polar coordinates, the wave function is

$$\psi = K(x + y + 2z)e^{-\alpha r}$$

$$= kr(\cos\varphi \sin\theta + \sin\varphi \sin\theta + 2\cos\theta)\, e^{-\alpha r}$$

$$= kr\left[\frac{1}{2}(e^{i\varphi} + e^{-i\varphi})\sin\theta + \frac{1}{2i}(e^{i\varphi} - e^{-i\varphi})\sin\theta \right.$$
$$\left. + 2\cos\theta\right] e^{-\alpha r}$$

$$= kr\left\{\left(\frac{1}{2} + \frac{1}{2i}\right)\sin\theta\, e^{i\varphi} + \left(\frac{1}{2} - \frac{1}{2i}\right)\sin\theta\, e^{-i\varphi} + 2\cos\theta\right\} e^{-\alpha r}$$

$$= kr e^{-\alpha r}\left[-\frac{1}{2}(1-i)\sqrt{\frac{8\pi}{3}}\, Y_1^1 + \frac{1}{2}(1+i)\sqrt{\frac{8\pi}{3}}\, Y_1^{-1} \right.$$
$$\left. + 2\sqrt{\frac{4\pi}{3}}\, Y_1^0\right]$$

so $l=1$, total angular momentum is

$$\sqrt{\langle L\rangle^2} = \sqrt{l(l+1)}\,\hbar = \sqrt{2}\,\hbar$$

b) The angular part of the wavefunction is

$$\psi = k'\left[-\frac{1}{2}(1-i)\sqrt{\frac{8\pi}{3}}\, Y_1^1 + \frac{1}{2}(1+i)\sqrt{\frac{8\pi}{3}}\, Y_1^{-1} + 2\sqrt{\frac{4\pi}{3}}\, Y_1^0\right]$$

The normalization condition gives

$$k'^2\left[\frac{1}{2}\frac{8\pi}{3} + \frac{1}{2}\frac{8\pi}{3} + 4\frac{4\pi}{3}\right] = 1$$

$$\Rightarrow k' = \sqrt{\frac{1}{8\pi}}$$

The z-component of angular momentum is

$$\langle\psi|L_z|\psi\rangle = k'^2\left\{\frac{1}{2}\frac{8\pi}{3}(+\hbar) + \frac{1}{2}\frac{8\pi}{3}(-\hbar)\right\} = 0$$

c) The probability of finding L_z to be $+\hbar$ is

$$P = |\langle L_z = +\hbar|\psi(\theta,\varphi)\rangle|^2$$

$$= \frac{1}{8\pi}\,\frac{1}{2}\,\frac{8\pi}{3} = \frac{1}{6}$$

d) The probability of finding the particle in the solid angle $d\Omega$ of θ, φ is

$$\int \psi^* (\theta, \varphi)\ \psi (\theta, \varphi)\ d\Omega$$

$$= \frac{1}{8\pi} \left[(\cos\varphi + \sin\varphi) \sin\theta + 2\cos\theta \right]^2 d\Omega.$$

QM A1.

(a) $\omega = \sqrt{\frac{g}{\ell}} \approx \sqrt{10 s^{-2}} \approx 3.15 s^{-1}$

$\hbar\omega \approx 10^{-34} J\cdot s \cdot 3.15 s^{-1} \approx 3\times 10^{-34} J \sim 2\times 10^{-15} eV$

can be completely ignored

(b)

The de Broglie wavelength

~~QMA~~

$$\lambda = \frac{h}{mv} \approx \frac{6.7\times 10^{-34}}{0.1\cdot 0.5} = 1.4\times 10^{-32} m$$

The diffraction angle

$$\theta = \frac{1.4\times 10^{-32} m}{1 m} = 1.4\times 10^{-32} rad$$

← negligible, no diffraction in any direction

QM A2

$$E = \frac{\hbar^2 k^2}{2m} \qquad (1)$$

$$J = \frac{\hbar}{m} \operatorname{Im}\left(\psi^* \frac{d\psi}{dx}\right) = \frac{\hbar k}{m} A \quad (2)$$

From (1)

$$k = \frac{\sqrt{2mE}}{\hbar} = \frac{\sqrt{2 \cdot 9.1 \times 10^{-31} kg \cdot 10 \cdot 1.6 \times 10^{-19} J}}{1.055 \times 10^{-34} J \cdot s}$$

$$= 16.2 \times 10^{9} m^{-1} = 16.2 \, nm^{-1}$$

$$v = \frac{\hbar k}{m} = \frac{1.055 \times 10^{-34}}{9.1 \times 10^{-31}} \cdot 16.2 \times 10^{9} = 1.88 \times 10^{6} \frac{m}{s}$$

From (2)

$$A = \frac{J}{v} = \frac{4 \times 10^{5} s^{-1}}{1.88 \times 10^{6} \frac{m}{s}} = 0.213 \, m^{-1}$$

QM

A3

Uncertainty principle

$$\Delta p \approx \frac{\hbar}{\Delta x} = \frac{\hbar}{D}$$

$$E_{min} = \frac{(\Delta p)^2}{2M_n} = \frac{\hbar^2}{2D^2 M_n} = \frac{(1.055\times10^{-34})^2}{2\times10^{-28}\cdot1.67\times10^{-27}}$$

$$(M_n \approx M_p) \quad = 0.33\times10^{-13}\,\mathrm{J} \approx 0.2\times10^{6}\,\mathrm{eV} = 0.2\,\mathrm{MeV}$$

A4

$$\sigma_i \psi = \lambda \psi$$

$$\sigma_i^2 \psi = \lambda \sigma_i \psi = \lambda^2 \psi$$

Since $\sigma_i^2 = I$, $\lambda = \pm 1$, therefore eigenvalues of σ_i are ± 1.

In the diagonal representation

$$\sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Tr(\sigma_i) = 0$$

Since Trace does not depend on representation,

$$Tr(U^{+}\sigma_i U) = Tr(\sigma_i) = 0$$

QM

B2

(a) $$\int |\psi|^2 dx = \int_{-b}^{b} K^2(b^2-x^2)\,dx = K^2\left(b^2x-\frac{x^3}{3}\right)\Big|_{-b}^{b}$$

$$= \frac{4}{3}K^2b^3 = 1$$

$$K = \frac{\sqrt{3}}{2b^{3/2}}$$

$$\cancel{(b)\quad \langle x\rangle = \int_{-\frac{b}{2}}^{0} K^2x(b^2-x^2)\,dx = K^2\left(\frac{1}{2}b^2x^2-\frac{1}{4}x^4\right)\Big|_{-\frac{b}{2}}^{0}}$$

$$\cancel{= K^2\Big(\frac{b^4}{8}}$$

(b) $$P = \int_{-b/2}^{0} K^2(b^2-x^2)\,dx = K^2\left(b^2x-\frac{x^3}{3}\right)\Big|_{-\frac{b}{2}}^{0}$$

$$= K^2\left(\frac{b^3}{2}-\frac{b^3}{24}\right) = K^2\cdot\frac{11}{24}b^3 = \frac{3}{4}\cdot\frac{11}{24} = \frac{11}{32} = 0.344$$

(c) $$\langle x\rangle = \int_{-b}^{b} K^2x(b^2-x^2)\,dx = 0$$

since integrand is an odd function

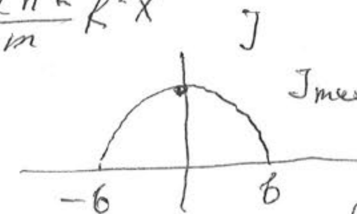
(d) $$\frac{dP}{dt}+\frac{dJ}{dx}=0$$

$k = $ ...

$$J = \frac{\hbar}{m}\,\mathrm{Im}\left(\psi^*\frac{d\psi}{dx}\right) = \frac{\hbar}{m}\,\mathrm{Im}\left[ik|\psi|^2-K^2x\right] = \frac{\hbar k}{m}|\psi|^2$$

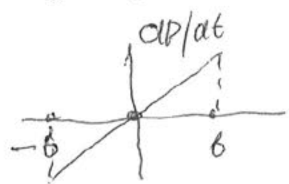
$$J = \frac{\hbar k}{m}K^2(b^2-x^2) \qquad \frac{dJ}{dx} = -\frac{2\hbar k}{m}K^2x$$

$$\frac{dP}{dt} = \frac{2\hbar k}{m}K^2x = \frac{3\hbar k}{2mb^3}\,x$$



$J_{max} = \frac{3\hbar k}{4mb}$

k ...



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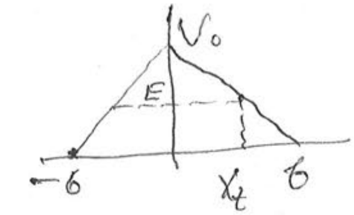
QM

B3

(a)



$$V = V_0\left(1-\frac{|x|}{b}\right)$$

turning point $E = V_0\left(1-\frac{x}{b}\right)$

$$x_t = b\left(1-\frac{E}{V_0}\right) \qquad E < V_0$$

$$D = \exp\left\{-\frac{2}{\hbar}\int_{-x_t}^{x_t}\left[2m(V-E)\right]^{1/2}dx\right] \qquad \int_{-x_t}^{x_t} = 2\int_0^{x_t}$$

$$\int_0^{x_t}\left[V_0\left(1-\frac{x}{b}\right)-E\right]^{1/2}dx = \frac{b}{V_0}\int_0^{\frac{V_0 x_t}{b}}(V_0-E-z)^{1/2}dz$$

$$z = \frac{V_0 x}{b} \quad \Bigg| = -\frac{b}{V_0}\cdot\frac{2}{3}\left(V_0-E-z\right)^{3/2}\Bigg|_0^{V_0x_t/b} = \frac{2}{3}\frac{b}{V_0}\left(V_0-E\right)^{3/2}$$

$$D = \exp\left[-\frac{8b\,(2m)^{1/2}}{3\hbar V_0}\left(V_0-E\right)^{3/2}\right]$$

(b) $\int_0^b (V_0-E)^{1/2}dx = (V_0-E)^{1/2}b$

$$D = \exp\left[-\frac{4}{\hbar}\left[2m(V_0-E)\right]^{1/2}b\right]$$

(c)

E	exponent (a)	(b)
0	$\frac{8b}{3\hbar}(2mV_0)^{1/2}$	$\frac{4b}{\hbar}(2mV_0)^{1/2}$
$\frac{V_0}{2}$	$\frac{4b}{3\hbar}(mV_0)^{1/2}$	$\frac{4b}{\hbar}(mV_0)^{1/2}$

in case (b) exponent is larger → probability is lower because barrier is thicker

for $E = V_0$ $D = 1$ in both cases. This is not correct because of the WKB (quasiclassical) approximation.

QM

B4

(a) $n=1$: no since it is a superposition of energy eigenstates

$n=2$: yes since it is an energy eigenstate

(b) ~~n=1~~

$$\langle E\rangle = \frac{1}{6}\left[2\frac{E_1}{n^2} + \frac{E_1}{4} + 3\frac{E_1}{4}\right] = \frac{E_1}{6}\left(\frac{2}{n^2}+1\right)$$

where $E_1 = R_y$ is the Rydberg unit of energy

$n=1$: $\langle E\rangle = \frac{E_1}{2}$

$n=2$: $\langle E\rangle = \frac{E_1}{4}$

(c) $\langle \vec{L}^2\rangle = \frac{1}{6}\left[2\cdot 0 + 1\cdot 2 + 3\cdot 2\right]\hbar^2 = \frac{7}{6}\hbar^2$ we use $\vec{L}^2 = \hbar^2 l(l+1)$

(d) $\langle L_z^2\rangle = \frac{1}{6}\{2\cdot 0 + 0 + 3\cdot 1\}\hbar^2 = \frac{1}{2}\hbar^2$ we use $L_z^2 = m^2\hbar^2$

(e) due to orthogonality of spherical harmonics; all crossed terms are 0, therefore

$$\langle r\rangle = \frac{1}{6}\left[2\langle r_{1s}\rangle + \langle r_{2p}\rangle + 3\langle r_{2p}\rangle\right] = \frac{1}{3}\left[\langle r_{1s}\rangle + 2\langle r_{2p}\rangle\right]$$

$$\langle r_{nl}\rangle = \int_0^\infty r^3 R_{nl}\,dr \qquad R_{1s}^2 = \frac{4}{a_0^3}e^{-2r/a_0} = \frac{4}{a_0^3}e^{-2x}, \quad x = \frac{r}{a_0}$$

$$\langle r_{1s}\rangle = \frac{4}{a_0^3}\int r^3 e^{-2x}\,dr = 4a_0\int_0^\infty x^3 e^{-2x}\,dx = 4a_0\frac{3!}{16} = \frac{3}{2}a_0$$

$$R_{2p}^2 = \frac{r^2}{24a_0^5}e^{-x} \qquad \langle r_{2p}\rangle = \frac{1}{24a_0^5}\int r^5 e^{-x}\,dx = \frac{a_0}{24}5! = 5a_0$$

$$\langle r\rangle = \frac{1}{3}\left(\frac{3}{2}a_0 + 10a_0\right) = \frac{23}{6}a_0$$