UNL - Department of Physics and Astronomy

Preliminary Examination - Day 2
Wednesday, May 29, 2024

This test covers the topics of *Quantum Mechanics* (Topic 1) and *Electrodynamics* (Topic 2). Each topic has 4 "A" questions and 4 "B" questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Quantum Mechanics Group A

Answer only two Group A questions

A1. Quantum phenomena are often negligible in the macroscopic world. Show this numerically for the following cases:

(a) The quantum of energy for a simple pendulum of length 1 m.

(b) The diffraction of a tennis ball of mass m = 0.1 kg moving at a speed v = 0.5 m/sec by a window of size 1×1.5 m^2.

A2. A beam of electrons has kinetic energy E =10 eV and the probability current J= 4.0×10^5 s^{-1}. The wavefunction of the electrons in the beam is described by $\psi = Ae^{ikx}$. Determine the quantities A and k , making sure to include units in your answers.

A3. The nucleus of the krypton atom has diameter D = 1.0×10^{-14} m. Use this information to estimate the minimum kinetic energy of a neutron in this nucleus (in eV).

A4. Consider three Pauli 2×2 matrices, σ_1, σ_2, and σ_3 which satisfy $\{\sigma_i, \sigma_j\}= 2\delta_{ij}$. where {} denotes the anticommutator. Without specifying the representation for the matrices, show that their eigenvalues are ±1 and that $\mathrm{Tr}(\sigma_i) = 0$.

Quantum Mechanics Group B

Answer only two Group B questions

B1. Consider a spinless particle represented by the wave function

$$\psi = K(x+y+2z)e^{-\alpha r},$$

where r is the radial coordinate, K and α are real constants.

(a) What is the total angular momentum of the particle?

(b) What is the expectation value of the z-component of angular momentum?

(c) If the z-component of angular momentum, L_z, were measured, find the probability thatthe result would be

$$L_z = +\hbar$$

(d) What is the probability of finding the particle at (θ, φ) and in solid angle $d\Omega$? Here (θ, φ) are the usual angles of spherical coordinates.

B2. The wavefunction of a particle of mass m moving along the x direction is given, at some time, by

$$\psi(x) = \begin{cases} Ke^{ipx/\hbar}\sqrt{b^2 - x^2} & \text{for } |b| < x \\ 0 & \text{elsewhere} \end{cases}$$

in which p and the normalization constant K are positive real numbers.

(a) Calculate the normalization constant K. What are the units of K?

(b) For the given wavefunction, you measure the particle's position. Calculate the probability that you find it between $x = -b/2$ and $x = 0$.

(c) Give a symmetry argument to explain why $\langle x \rangle = 0$.

(d) Calculate the current j (probability flux per second) as a function of position x.

(e) Using the continuity equation, calculate the time derivative of the probability density dP/dt as a function of x. Sketch j and dP/dt as functions of x.

B3. Using the WKB approximation, **f**ind the tunneling probability for a particle of mass m and energy E passing through

(a) the triangular barrier of the form

$$V = \begin{cases} V_0\left(1-|x|/b\right) & \text{for } |x| < b, \\ 0 & \text{otherwise.} \end{cases}$$

(b) the rectangular barrier

$$V = \begin{cases} V_0 & \text{for } |x| < b, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Calculate your answers for a very small $E << V_0$, and $E = V_0/2$, and explain the difference between cases (a) and (b). What do your answers say for $E=V_0$? Is this result correct? Why?

B4. An electron in the Coulomb field of a proton is in a state described by the wave function

$$\psi(\mathbf{r}) = \frac{1}{\sqrt{6}}\left[\sqrt{2}\psi_{n00}(\mathbf{r}) - \psi_{210}(\mathbf{r}) + \sqrt{3}\psi_{211}(\mathbf{r})\right].$$

(a) Is this state an eigenstate of the Hamiltonian if $n = 1$? If $n = 2$? Explain.

(b) Find the expectation value of the energy for $n = 1$ and $n = 2$. Express your answer in terms of the ground-state energy of the hydrogen atom.

(c) Find the expectation value of $\mathbf{L}^2$.

(d) Find the expectation value of L_z^2.

(e) Find the expectation value of the radial coordinate r for the state above if $n = 1$.

Electrodynamics Group A

Answer only two Group A questions

A1. A linear homogeneous dielectric of arbitrary shape is placed in a static electric field. There is no free charge density in the dielectric. The dielectric has no spatial dispersion, i.e., the electric displacement **D** and the electric field **E** are related by a local dielectric permittivity: $\mathbf{D}(\mathbf{r}) = \varepsilon \mathbf{E}(\mathbf{r})$. Show that the polarization charge can appear only on the surface of this dielectric but not in the bulk. Now, assume that the dielectric is inhomogeneous, i.e., its dielectric permittivity $\varepsilon(\mathbf{r})$ is not uniform. Write the expression for the volume bound charge density in terms of $\varepsilon(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$.

A2. Find the magnetic moment of a thin spherical shell of radius R that carries a constant surface charge density σ and rotates with frequency ω about an axis passing through its center.

A3. A parallel-plate capacitor has circular plates of radius a, each being charged with the rate k (given in units of C m^{-2} s^{-1}) so that the surface charge density on the plates changes in time as $\sigma(t) = \pm(\sigma_0 + kt)$, where signs + and – correspond to the top and bottom plates, respectively.

(a) Find the electric field between the plates. Neglect the contribution to the electric field from the Faraday's law.

(b) Find the magnetic field at the distance r from the symmetry axis, assuming $r << a$.

A4. A plane electromagnetic wave propagating in the z direction is described by the following scalar and vector potentials

$$\Phi = 0,$$
$$\mathbf{A} = A_0 \hat{\mathbf{x}} \sin(kz - \omega t).$$

Consider a gauge transformation with the gauge function

$$\lambda = \lambda_0 \cos(kz - \omega t).$$

(a) Obtain the potentials in the new gauge.

(b) Do the new potentials satisfy the Coulomb gauge condition?

(c) Show that the new potentials satisfy the Lorenz gauge condition.

Electrodynamics Group B

Answer only two Group B questions

B1. A slab of uniform volume charge density ρ_0 has its bottom and top surfaces at $z = -d$ and $z = 0$, respectively. Another slab of uniform volume charge density $-\rho_0$ is placed above, so that its bottom and top surfaces are at $z = 0$ and $z = d$, respectively. Find the electric field and electrostatic potential in all space. Sketch the electric field and the electrostatic potential as a function of z.

B2. Perfect conductors completely screen magnetic fields in their interior by creating persistent surface currents. Consider a semi-infinite perfect conductor occupying volume $z < 0$ and having a flat surface at $z = 0$. An infinitely long straight wire is suspended at $z = d$ above the conductor surface and carries current I in the x direction. Find the surface current density $\mathbf{K}$ induced on the perfect conductor's surface. Show that the total current on the surface is equal in magnitude to the current in the wire but flows in the opposite direction.

B3. A disk of radius R has a uniform surface charge density σ and is rotated with angular frequency ω around the axis z perpendicular to the disk and passing through its center. Find the magnetic field $\mathbf{B}$ on the axis z.

B4. An electrical circuit consists of a long thin cylindrical conducting shell of radius R and a parallel return wire of radius r on the axis of the shell. The current is assumed to be distributed uniformly over the cross section of the wire. Calculate the self-inductance of this circuit per unit length. Assume that the permeability of the conductor is μ_0.

Physical constants

Speed of light .... $c = 2.998 \times 10^8$ m/s

Planck's constant .... $h = 6.626 \times 10^{-34}$ J$\cdot$s

Planck's constant / 2π.... $\hbar = 1.055 \times 10^{-34}$ J$\cdot$s

Electron mass .... $m_e = 9.109 \times 10^{-31}$ kg

Electron's rest energy .... 511.0 keV

Boltzmann constant .... $k_\mathrm{B} = 1.381 \times 10^{-23}$ J/K

Compton wavelength ..... $\lambda_\mathrm{C} = \dfrac{h}{m_\mathrm{e} c} = 2.426$ pm

Elementary charge .... $e = 1.602 \times 10^{-19}$ C

Proton mass .... $m_p = 1.673 \times 10^{-27}\,\mathrm{kg} = 1836\, m_e$

Electric permittivity ...... $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

Bohr radius.................... $a_0 = \dfrac{4\pi\varepsilon_0 \hbar^2}{e^2 m_\mathrm{e}} = 0.5292$ Å

Magnetic permeability.... $\mu_0 = 1.257 \times 10^{-6}$ H/m

1 rydberg .... $Ry = 13.6$ eV

1 hartree (= 2 Ry)............ $E_h = \dfrac{\hbar^2}{m_e a_0^2} = 27.21$ eV

Molar gas constant.......... $R = 8.314$ J / mol$\cdot$K

Gravitational constant...... $G = 6.674 \times 10^{-11}$ m^3/ kg s^2

Avogadro constant .... $N_\mathrm{A} = 6.022 \times 10^{23}$ mol^{-1}

hc .... $hc = 1240$ eV$\cdot$nm

Fine structure constant .. $\alpha = \dfrac{1}{4\pi\varepsilon_0} \dfrac{e^2}{\hbar c}$

Neutron mass .... 1.7×10^{-27} kg

Equations That May Be Helpful

TRIGONOMETRY

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$\sin(2\theta)=2\sin\theta\cos\theta$$
$$\cos(2\theta)=\cos^2\theta-\sin^2\theta=1-2\sin^2\theta=2\cos^2\theta-1$$
$$\sin\alpha\sin\beta=\tfrac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta=\tfrac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta=\tfrac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta=\tfrac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

$$\cos(ix)=\cosh(x)$$
$$\sin(ix)=i\sinh(x)$$

For small x:

$$\sin x\approx x-\tfrac{1}{6}x^3$$
$$\cos x\approx 1-\tfrac{1}{2}x^2$$
$$\tan x\approx x+\tfrac{1}{3}x^3$$

QUANTUM MECHANICS

$$\left[AB,C\right]=A\left[B,C\right]+\left[A,C\right]B$$

Angular momentum: $[L_x,L_y]=i\hbar L_z \quad et\ cycl.$

Ladder operators:
$$L_+\,|\,\ell,m\rangle=\hbar\sqrt{(\ell+m+1)(\ell-m)}\,|\,\ell,m+1\rangle$$
$$L_-\,|\,\ell,m\rangle=\hbar\sqrt{(\ell+m)(\ell-m+1)}\,|\,\ell,m-1\rangle$$

Gyromagneic ratio for electron (SI units) = e/m

Pauli matrices: $\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Barrier penetration in the WKB approximation: $P=\exp\left\{-\dfrac{2}{\hbar}\int\sqrt{2m\left[V(x)-E\right]}dx\right\}$

Table Spherical harmonics and their expressions in Cartesian coordinates.

$Y_{lm}(\theta, \varphi)$	$Y_{lm}(x, y, z)$
$Y_{00}(\theta, \varphi) = \frac{1}{\sqrt{4\pi}}$	$Y_{00}(x, y, z) = \frac{1}{\sqrt{4\pi}}$
$Y_{10}(\theta, \varphi) = \sqrt{\frac{3}{4\pi}} \cos\theta$	$Y_{10}(x, y, z) = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$
$Y_{1,\pm 1}(\theta, \varphi) = \mp\sqrt{\frac{3}{8\pi}} e^{\pm i\varphi} \sin\theta$	$Y_{1,\pm 1}(x, y, z) = \mp\sqrt{\frac{3}{8\pi}} \frac{x \pm iy}{r}$
$Y_{20}(\theta, \varphi) = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$	$Y_{20}(x, y, z) = \sqrt{\frac{5}{16\pi}} \frac{3z^2 - r^2}{r^2}$
$Y_{2,\pm 1}(\theta, \varphi) = \mp\sqrt{\frac{15}{8\pi}} e^{\pm i\varphi} \sin\theta\cos\theta$	$Y_{2,\pm 1}(x, y, z) = \mp\sqrt{\frac{15}{8\pi}} \frac{(x \pm iy)z}{r^2}$
$Y_{2,\pm 2}(\theta, \varphi) = \sqrt{\frac{15}{32\pi}} e^{\pm 2i\varphi} \sin^2\theta$	$Y_{2,\pm 2}(x, y, z) = \mp\sqrt{\frac{15}{32\pi}} \frac{x^2 - y^2 \pm 2ixy}{r^2}$

Stationary states of harmonic oscillator for $n = 0$ and $n = 1$:

$$\varphi_0(x) = \left(\frac{\alpha}{\pi^{1/2}}\right)^{1/2} \mathrm{e}^{-\frac{\alpha^2 x^2}{2}},$$

$$\varphi_1(x) = \left(\frac{\alpha}{\pi^{1/2}}\right)^{1/2} 2\alpha x\, \mathrm{e}^{-\frac{\alpha^2 x^2}{2}},$$

where $\alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2}$.

Hydrogen atom: $E_n = -\frac{Ry}{n^2}$, $Ry = \frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}$

Radial functions for the hydrogen atom $R_{nl}(r)$:

$$R_{10}(r) = \frac{2}{a_0^{3/2}} \exp\left(-\frac{r}{a_0}\right),$$

$$R_{20}(r) = \frac{2}{(2a_0)^{3/2}} \left[1 - \frac{r}{2a_0}\right] \exp\left(-\frac{r}{2a_0}\right),$$

$$R_{21}(r) = \frac{r}{24^{1/2} a_0^{5/2}} \exp\left(-\frac{r}{2a_0}\right).$$

ELECTROSTATICS

$$\oiint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{q_{\text{encl}}}{\varepsilon_0}; \qquad \mathbf{E} = -\nabla\Phi; \qquad \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{E}\cdot d\boldsymbol{\ell} = \Phi(\mathbf{r}_1) - \Phi(\mathbf{r}_2); \qquad \Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

Work done: $W = -\int_{\mathbf{a}}^{\mathbf{b}} q\mathbf{E}\cdot d\boldsymbol{\ell} = q\left[\Phi(\mathbf{b}) - \Phi(\mathbf{a})\right]$.

Energy stored in electric field: $W = \frac{1}{2}\varepsilon_0 \int_V E^2 d\tau = Q^2/2C$.

Multipole expansion: $\Phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r} + \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}\cdot\mathbf{p}}{r^3} + \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_{ij} Q_{ij}\frac{x_i x_j}{r^5} + \ldots$

Monopole moment: $q = \int \rho(\mathbf{r}) d^3\mathbf{r}$.

Dipole moment: $\mathbf{p} = \int \rho(\mathbf{r})\mathbf{r}\, d^3\mathbf{r}$.

Quadrupole moment : $Q_{ij} = \int \rho(\mathbf{r})\left[3r_i r_j - r^2\delta_{ij}\right] d^3\mathbf{r}$ (notation: $r_1 = x,\ r_2 = y,\ r_3 = z$).

Parallel-plate capacitor: $C = \varepsilon_0 \frac{A}{d}$.

Spherical capacitor: $C = 4\pi\varepsilon_0 \frac{ab}{b-a}$.

Cylindrical capacitor: $C = 2\pi\varepsilon_0 \frac{L}{\ln(b/a)}$ (for a length L).

Relative permittivity: $\varepsilon_{\text{r}} = 1 + \chi_{\text{e}}$.

Bound charges: $\rho_{\text{b}} = -\nabla\cdot\mathbf{P}$; $\sigma_{\text{b}} = \mathbf{P}\cdot\hat{\mathbf{n}}$.

MAGNETOSTATICS

Relative permeability: $\mu_{\text{r}} = 1 + \chi_{\text{m}}$.

Lorentz force: $\mathbf{F} = q\mathbf{E} + q(\mathbf{v}\times\mathbf{B})$.

Current densities: $I = \int \mathbf{J}\cdot d\mathbf{A}$, $I = \int \mathbf{K}\cdot d\boldsymbol{\ell}$.

Biot-Savart Law: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{I d\boldsymbol{\ell}\times\hat{\mathbf{R}}}{R^2}$ ($\mathbf{R}$ is vector from source point to field point $\mathbf{r}$).

B-field inside of an infinitely long solenoid: $\mathbf{B} = \mu_0 n I \hat{\boldsymbol{\varphi}}$ (n is the number of turns per unit length).

Ampere's law: $\oint \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 I_{\text{encl}}$.

Magnetic dipole moment of a planar current distribution: $\mathbf{m} = I\int d\mathbf{a}$.

Force on a magnetic dipole: $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$.

Torque on a magnetic dipole: $\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}$.

B-field of magnetic dipole: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\dfrac{3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}})-\mathbf{m}}{r^3}$.

Bound currents: $J_\mathrm{b} = \nabla\times\mathbf{M}$; $K_\mathrm{b} = \mathbf{M}\times\hat{\mathbf{n}}$.

<u>Maxwell's equations in vacuum</u>

$\nabla\cdot\mathbf{E} = \dfrac{\rho}{\varepsilon_0}$	Gauss' law
$\nabla\cdot\mathbf{B} = 0$	no magnetic charge
$\nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}$	Faraday's law
$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \varepsilon_0\mu_0\dfrac{\partial\mathbf{E}}{\partial t}$	Ampere's law with Maxwell's correction

<u>Maxwell's equations in linear, isotropic, and homogeneous media</u>

$\nabla\cdot\mathbf{D} = \rho_\mathrm{f}$	Gauss' law
$\nabla\cdot\mathbf{B} = 0$	no magnetic charge
$\nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}$	Faraday's law
$\nabla\times\mathbf{H} = \mathbf{J}_\mathrm{f} + \dfrac{\partial\mathbf{D}}{\partial t}$	Ampere's law with Maxwell's correction

Alternative way of writing Faraday's law: $\oint \mathbf{E}\cdot d\boldsymbol{\ell} = -\dfrac{d\mathsf{F}_B}{dt}$.

Mutual and self inductance: $\mathsf{F}_2 = M_{21}I_1$; $\mathsf{F} = LI$.

Energy stored in magnetic field: $W = \tfrac{1}{2}\mu_0^{-1}\int_V B^2 d\tau = \tfrac{1}{2}LI^2 = \tfrac{1}{2}\oint \mathbf{A}\cdot\mathbf{I}\,d\ell$.

Wave equations in a conducting medium:

$$\nabla^2\mathbf{E} = \mu\sigma\frac{\partial\mathbf{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}; \quad \nabla^2\mathbf{B} = \mu\sigma\frac{\partial\mathbf{B}}{\partial t} + \mu\varepsilon\frac{\partial^2\mathbf{B}}{\partial t^2}.$$

Gauge transformation:

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda; \quad \Phi' = \Phi - \frac{\partial\Lambda}{\partial t}.$$

Coulomb gauge:

$$\nabla\cdot\mathbf{A} = 0.$$

Lorenz gauge:

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0.$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$$\int_0^\infty \frac{1}{1+bx^2}dx = \frac{\pi}{2b^{1/2}}$$
$$\int_0^\infty x^n e^{-bx}dx = \frac{n!}{b^{n+1}}$$
$$\int (x^2+b^2)^{-1/2}dx = \ln\left(x+\sqrt{x^2+b^2}\right)$$
$$\int (x^2+b^2)^{-1}dx = \frac{1}{b}\arctan\left(\frac{x}{b}\right)$$
$$\int (x^2+b^2)^{-3/2}dx = \frac{x}{b^2\sqrt{x^2+b^2}}$$
$$\int (x^2+b^2)^{-2}dx = \frac{\dfrac{bx}{x^2+b^2}+\arctan\left(\dfrac{x}{b}\right)}{2b^3}$$
$$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$$
$$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$$
$$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) = -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$$
$$\int x^4e^{-x}dx = -e^{-x}\left(x^4+4x^3+12x^2+24x+24\right)$$
$$\int_0^\infty x^n e^{-x}dx = n!$$

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$$
$$\int_0^\infty xe^{-x^2}dx = \frac{1}{2a}$$
$$\int_0^\infty x^2e^{-x^2}dx = \frac{\sqrt{\pi}}{2a^{3/2}}$$
$$\int_0^\infty x^3e^{-x^2}dx = \frac{1}{2a^2}$$
$$\int_0^\infty x^4e^{-x^2}dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$$
$$\int_0^\infty x^5e^{-x^2}dx = \frac{1}{a^3}$$
$$\int_0^\infty x^6e^{-x^2}dx = \frac{15\sqrt{\pi}}{16a^{7/2}}$$