Mech A1

(a) $$m\frac{dv}{dt} = -Cv^2$$

$$\frac{dv}{v^2} = -\frac{C}{m}dt$$

$$\frac{1}{v_0} - \frac{1}{v} = -\frac{C}{m}t$$

$$v = \frac{v_0}{1+\frac{v_0 C}{m}t}$$

if $v = \frac{v_0}{2}$, $1+\frac{v_0 C}{m}t = 2$

$$t = \frac{m}{v_0 C} = \frac{5\times10^3\,kg}{10^4\,\frac{m}{s}\cdot 10^{-5}\,\frac{kg}{m}} = 5\times10^4\,s = 1.4\,h$$

(b) integrate now

$$\frac{dx}{dt} = \frac{v_0}{1+\frac{v_0 C}{m}t}$$

$$x = v_0\int_0^t \frac{dt}{1+\frac{v_0 C}{m}t} = \frac{m}{C}\ln\left(1+\frac{v_0 C}{m}t\right)$$

for $t = \frac{m}{v_0 C}$ $\quad x = \frac{m}{C}\ln 2 = 5\times10^8\,m\,\ln 2 = 3.47\times10^8\,m$

A2

$$\frac{mv^2}{R} = \mu mg \Rightarrow v = \sqrt{\mu R g}$$

$x = vt$, where $h = \frac{1}{2} g t^2 \Rightarrow t = \sqrt{2h/g}$

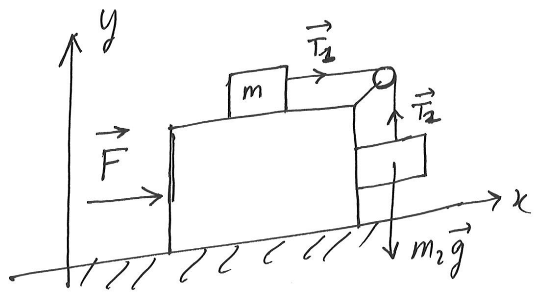
$$\Rightarrow x = \sqrt{\mu R g}\sqrt{\frac{2h}{g}}$$

$\boxed{x = \sqrt{2\mu R h}}$ is independent of m, and g.

A3

$(M+m_1+m_2)\,\ddot{x} = F$ (1)

because the accelerations of m_1, m_2 and M are the same when there is no relative motion among them.



$m_1\ddot{x} = T_1$ (2)

As there is no relative motion of m_2 along the x-axis:

$T_2 = m_2 g$ (3)

pulley radius

mass of the pulley is negligible: $T_1 R - T_2 R = I_{pulley}\,\alpha \approx 0$

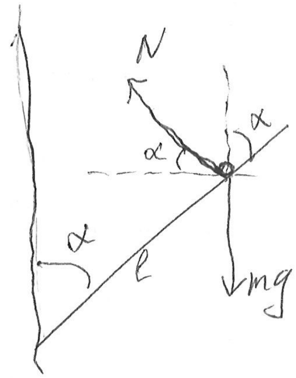
$T_1 = T_2$

(2) $\Rightarrow$ $\ddot{x} = \frac{T_1}{m_1} = \frac{T_2}{m_1} = \frac{m_2 g}{m_1}$ (4)

(4) in (1) $\Rightarrow$ $\boxed{F = \frac{m_2(M+m_1+m_2)}{m_1}\,g}$

Mech

A4



$N \sin\alpha = mg$

$N \cos\alpha = m\omega^2 l \sin\alpha$

eliminate N

$$\cot\alpha = \frac{\omega^2 l \sin\alpha}{g}$$

$\omega \cdot l \sin\alpha = v$

$$\omega^2 = \frac{g \cot\alpha}{l \sin\alpha} \qquad v^2 = \omega^2 l^2 \sin^2\alpha = g l \cos\alpha$$

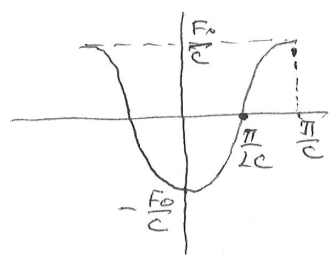
$$l = \frac{v^2}{g \cos\alpha}$$

Tuan

B.1

(a) $V(x) = +F_0 \int \sin(cx)\,dx = -\frac{F_0}{c}\cos(cx)$

if we choose integration constant 0, then $V(0) = -\frac{F_0}{c}$



(b) $\frac{mv_0^2}{2} - \frac{F_0}{c} = \frac{mv^2}{2} - \frac{F_0}{c}\cos(cx)$

$v = \left[v_0^2 + \frac{2F_0}{mc}(\cos cx - 1)\right]^{1/2}$

(c) motion is confined if $E < \frac{F_0}{c}$

$\frac{mv_0^2}{2} - \frac{F_0}{c} < \frac{F_0}{c}$

$v_0^2 < \frac{4F_0}{mc}$ or $v_0 < 2\sqrt{\frac{F_0}{mc}}$

turning ports: $E = V(x)$

$\frac{mv_0^2}{2} - \frac{F_0}{c} = -\frac{F_0}{c}\cos cx$

$x = \pm\frac{1}{c}\cos^{-1}\left(1 - \frac{cmv_0^2}{2F_0}\right)$

(d) $F \approx -F_0 cx$

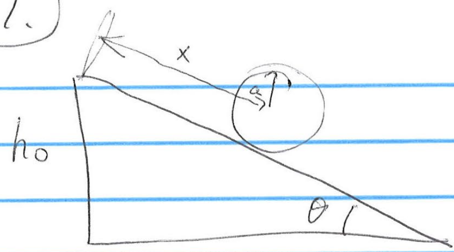
periodic motion with the force constant $k = F_0 c$

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{F_0 c}{m}}$ $T = 2\pi\sqrt{\frac{m}{F_0 c}}$

$v_0 = A\omega_0 \rightarrow A = \frac{v_0}{\omega_0} = v_0\sqrt{\frac{m}{F_0 c}}$

7.

Mech B2.



(a) $h = h_0 - x\sin\theta \qquad V = mg(h_0 - x\sin\theta)$

$$T = \frac{m\dot{x}^2}{2} + \frac{I\dot{\omega}^2}{2} = \frac{m\dot{x}^2}{2} + \frac{I\dot{x}^2}{2a^2} \qquad \dot{x} = a\dot{\omega}$$

$$I = \tfrac{1}{2}ma^2$$

$$L = \frac{1}{2}\left(m + \frac{I}{a^2}\right)\dot{x}^2 - mg(h_0 - x\sin\theta)$$ 10

(b) $$\frac{\partial L}{\partial \dot{x}} = \left(m + \frac{I}{a^2}\right)\dot{x} = \frac{3}{2}m\dot{x}$$

$$\frac{\partial L}{\partial x} = mg\sin\theta$$

$$\frac{3}{2}m\ddot{x} - mg\sin\theta = 0 \quad (1)$$ 10

$$\ddot{x} = \frac{2}{3}g\sin\theta$$

(c) $$p = \frac{\partial L}{\partial \dot{x}} = \frac{3}{2}m\dot{x}$$

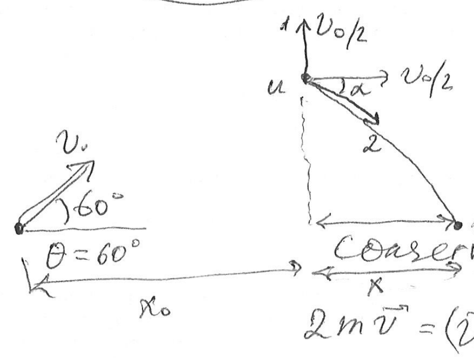
$$H = \frac{1}{2}\left(\frac{3}{2}m\right)\dot{x}^2 + mg(h_0 - x\sin\theta) = \frac{1}{2}\frac{p^2}{\frac{3}{2}m} + mg(h_0 - x\sin\theta)$$

$$\dot{x} = \frac{\partial H}{\partial p} \rightarrow \dot{x} = \frac{2}{3}\frac{p}{m}$$ 10

$$\dot{p} = -\frac{\partial H}{\partial x} \rightarrow \dot{p} = mg\sin\theta$$

substituting $\dot{p} = \frac{3}{2}m\ddot{x}$, we obtain Eq.(1)

Mech B3



the speed of the shell at the upper point is $v_0 \cos 60° = \frac{v_0}{2}$

Conservation of momentum

$2m\vec{v} = (\vec{v}_1 + \vec{v}_2)m$ where m is the mass of each fragment

for x and y components we have

$$\vec{v} = \left(\frac{v_0}{2}, 0\right) \quad \vec{v}_1 = \left(0, \frac{v_0}{2}\right)$$

$$\vec{v}_2 = 2\vec{v} - \vec{v}_1 = \left(v_0, -\frac{v_0}{2}\right)$$

$$|\vec{v}_2| = v_0\left(1 + \frac{1}{4}\right)^{1/2} = 1.118\, v_0$$

$$\tan\alpha = -\frac{1}{2} \rightarrow \alpha = -26.6°$$

Solve for the distance x between the upper point u and point l

$$y = x\tan\alpha - \frac{g x^2}{2 v_2^2 \cos^2\alpha} = -h \quad (1)$$

where $h = \frac{v_0^2 \sin^2 60°}{2g} = \frac{10^4 \cdot \frac{3}{4}}{20} = 375\,m \quad (g \approx 10\,\frac{m}{s^2})$

$\frac{1}{\cos^2\alpha} = 1 + \tan^2\alpha = \frac{5}{4}$

Substitute into (1)

$$-\frac{x}{2} - \frac{5x^2}{10^4} \cdot \frac{5}{4} = -375$$

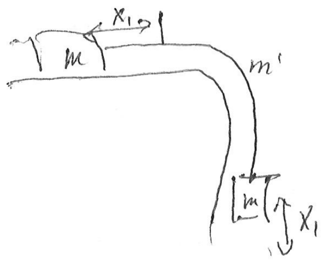
$$25x^2 + 2\times 10^4 x - 15\times 10^6 = 0$$

$$\rightarrow x = 471\,m$$

The distance between the launching point and the upper point is $\frac{v_0^2 \sin(2\cdot 60°)}{2g} = \frac{10^4 \frac{\sqrt{3}}{2}}{20} = 433\,m$

total horizontal distance $471 + 433 = 904\,m$

Mech B4



Choosing the generalized coordinate x_1, we have for the potential energy

$$V = -mgx_1 - \frac{m'}{l}x_1 g \frac{x_1}{2}$$

~~Since~~ x_1 is the part of the cord which has become vertical

Therefore the Lagrangian is

$$L = \left(m + \frac{m'}{2}\right)\dot{x}_1^2 + mgx_1 + \frac{m'}{l}g\frac{x_1^2}{2}$$

(b) Lagr equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0$$

$$(2m+m')\ddot{x}_1 - mg - \frac{m'g}{l}x = 0$$

$$\ddot{x}_1 = \frac{mg}{2m+m'} + \frac{m'g}{(2m+m')l}x$$

general solution

$$x_1 = C_1 \sinh\omega t + C_2\cosh\omega t - \frac{lm}{m'}, \quad \omega = \sqrt{\frac{m'g}{2m+m'}}$$

with the initial conditions

$$x_1(0) = 0, \quad \dot{x}_1(0) = 0$$

$$0 = C_2 - \frac{lm}{m'}, \quad 0 = C_1$$

$$x_1 = \frac{lm}{m'}(\cosh\omega t - 1)$$

$$\dot{x}_1 = \omega l\frac{m}{m'}\sinh\omega t$$

exponential form

$$x_1 = \frac{lm}{m'}\left[\frac{e^{\omega t}+e^{-\omega t}}{2} - 1\right]$$

$$\dot{x}_1 = \omega l\frac{m}{m'}\frac{e^{\omega t}-e^{-\omega t}}{2}$$

Note: add l in the denominator in the expression for ω

Preliminary Thermal – May 2024

Easy Problems:

1. In a vacuum tube of pressure 2×10^{-3} Pa, at 27 °C, calculate:
   a. number of gas particles per m^3,
   b. volume occupied per particle,
   c. mean free path of the particle (assuming the particle has a radius of 155 pm).

   Hint: You need to calculate the scattering cross-section and scattering volume. There is a factor of $\sqrt{2}$ when considering the relative motion between particles.

   Solution:

   a. Using PV=nRT or PV = Nk_BT, one can calculate the particle density N/V = P/k_BT = $4.83\times10^{17}/m^3$.
   b. Volume occupied per particle is: V/N=2.07×10^{-18} m^3.
   c. Scattering cross-section is $4\pi r^2$

   Average scattering volume is $4\pi r^2\lambda = \frac{V}{\sqrt{2}N}$. Here the mean free path is reduced by a factor of $\sqrt{2}$ compared to the static case due to the relative motion between two particles $|\overrightarrow{v_1} - \overrightarrow{v_2}| = (|\overrightarrow{v_1} - \overrightarrow{v_2}|^2)^{1/2} = (|\overrightarrow{v_1}|^2 - 2|\overrightarrow{v_1}||\overrightarrow{v_2}| + |\overrightarrow{v_2}|^2)^{1/2} = \sqrt{2}|\vec{v}|$.

   Mean free path is $\lambda = \frac{V}{4\pi\sqrt{2}r^2N}$=4.85 m.

2. Find the thermal expansion coefficient α=(∂V/∂*T*)$_P$/V, isothermal compressibility K_T=-(∂V/∂P)$_T$/V for ideal gas.

   Solution:

   For idea gas, V=nRT/P.

   So, $\left(\frac{\partial V}{\partial T}\right)_P = \frac{nR}{P}, \left(\frac{\partial V}{\partial P}\right)_T = -\frac{nRT}{P^2}$.

   $$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = \frac{nR}{PV} = \frac{1}{T},\ K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = -\frac{1}{V}\left(-\frac{nRT}{P^2}\right) = \frac{1}{P}.$$

3. A material's density is ρ_s and ρ_l for solid and liquid phase respectively. Given that the latent heat for the solid -> liquid transition is L/kg at pressure P, find the change of internal energy during the transition per kg.

   Solution:

   According to the first law, ΔU=Q-W.

   The work done by the material during the solid->liquid transition is:

$W=\int PdV = P\int dV= P\Delta V=P(1/\rho_l-1/\rho_s)$.

$\Delta U=Q-W=L- P(1/\rho_l-1/\rho_s)$.

A4

pb # 6 : Thermodynamics (easy)

a) at constant temperature T_0, the work is

$$W = \int_A^B p\,dV = RT_0 \int_{V_0}^{2V_0} \frac{dV}{V} = \underline{\underline{RT_0 \ln 2}}$$

As the change of the internal energy is zero,
the heat absorbed by the gas

$$\underline{\underline{Q = W = RT_0 \ln 2}}$$

b) At constant pressure p, the work is

$$W = \int_{V_0}^{2V_0} p\,dV = pV_0 = \underline{\underline{RT_0}}$$

the increase of the internal energy is

$$\Delta U = C_v \Delta T = \frac{3}{2} R\Delta T = \frac{3}{2} p\Delta V = \frac{3}{2} pV_0 = \frac{3}{2} RT_0$$

Thus the heat absorbed by the gas is

$$Q = \Delta U + W = \underline{\underline{\frac{5}{2} RT_0}}$$

Hard Problems:

1. A rigid adiabatic container is divided into two parts containing n_1 and n_2 mole of ideal gases respectively, by a movable and thermally conducting wall. Their pressure and volume are P_1, V_1 for part 1 and P_2, V_2 for part 2 respectively. Find the final pressure P and temperature T after the two gas reaches equilibrium. Assuming the constant volume specific heats of the two gas are the same.

   Solution:

   For the initial state $T_1= P_1V_1/n_1R$, $T_2= P_2V_2/n_2R$.

   After the two gas reaches equilibrium, their volumes can be assumed as V'_1 and V'_2.

   Since the internal energy does not change:

   $Cv\, n_1\, (T-T_1) = Cv\, n_2\, (T_2-T)$

   So, $T= (n_1\, T_1+n_2\, T_2)/(n_1+n_2) = (P_1V_1/R + P_2V_2/R)/(n_1+n_2)$

   The pressure of the two gas are the same. Therefore,

   $P = n_1RT/V'_1= n_2RT/V'_2$

   Hence, $P = (n_2+n_1)RT/(V'_1+V'_2)$

   Since $V_1 + V_2= V'_1 + V'_2$

   one has

   $P = (n_2+n_1)RT/(V_1+V_2) = (n_2+n_1)R/(V_1+V_2)\ (P_1V_1/R + P_2V_2/R)/(n_1+n_2)$

   $= (P_1V_1 + P_2V_2)\ /(V_1+V_2)$

2. In a throttling process, the Joule-Thompson coefficient is defined as $\mu= (\partial T/\partial P)_H$.

   a. Given the relation $(\partial H/\partial P)_T=V-T\ (\partial V/\partial T)_P$, show that $\mu=V(T\alpha-1)/C_P$, where $\alpha=(\partial V/\partial T)_P/V$. (Hint, use the cyclic rule)

   b. Show that for ideal gas, $\mu=0$.

   Solution:

   a. Using the cyclic rule, $\left(\frac{\partial T}{\partial P}\right)_H\left(\frac{\partial P}{\partial H}\right)_T\left(\frac{\partial H}{\partial T}\right)_P=-1.$

      Therefore, $\left(\frac{\partial T}{\partial P}\right)_H=\frac{1}{\left(\frac{\partial P}{\partial H}\right)_T\left(\frac{\partial H}{\partial T}\right)_P}=\frac{\left(\frac{\partial P}{\partial H}\right)_T}{\left(\frac{\partial H}{\partial T}\right)_P}.$

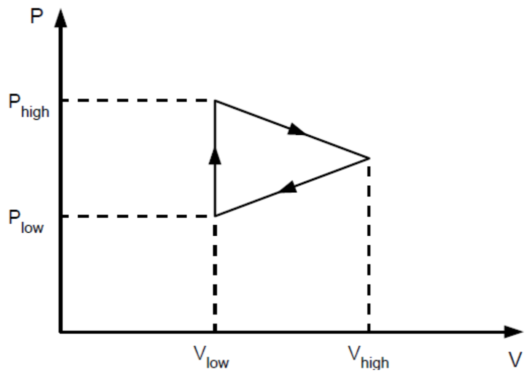
      Since $\left(\frac{\partial H}{\partial T}\right)_P=C_P$ and $\left(\frac{\partial H}{\partial P}\right)_T=V-T\left(\frac{\partial V}{\partial T}\right)_P$,

      $\left(\frac{\partial T}{\partial P}\right)_H=\frac{V-T\left(\frac{\partial V}{\partial T}\right)_P}{C_P}=\frac{V}{C_P}\left(1-\frac{T}{V}\left(\frac{\partial V}{\partial T}\right)_P\right)=\frac{V}{C_P}(1-T\alpha),$

      where $\alpha=\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P.$

   b. For ideal gas, $\alpha=\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P=\frac{1}{T}$. So, $\left(\frac{\partial T}{\partial P}\right)_H=\frac{V}{C_P}\left(1-\frac{T}{T}\right)=0$

3. The cycle of a hypothetical engine is illustrated below. Let $P_{low} = 1 \times 10^6$ Pa, $P_{high} = 2 \times 10^6$ Pa, $V_{low} = 5 \times 10^{-3} m^3$, and $V_{high} = 25 \times 10^{-3} m^3$. If the energy absorbed by heating the engine is 5×10^4 J, what is the efficiency of the engine? Calculate the highest and lowest temperature (T_h *and* T_c) in the cycle and calculate $1 - \frac{T_c}{T_h}$, assume that the pressure of the V_{high} point is $(P_{low}+P_{high})/2$.



Solution:

We construct the table:

	ΔU	$Q = \Delta U + W$	W
$1 \rightarrow 2$	$\frac{3}{2}(P_0V_h - P_hV_l) > 0$	> 0	$\frac{(P_h + P_0)(V_h - V_l)}{2} > 0$
$2 \rightarrow 3$	$\frac{3}{2}(P_lV_l - P_0V_h) < 0$	< 0	$-\frac{(P_l + P_0)(V_h - V_l)}{2} < 0$
$3 \rightarrow 1$	$\frac{3}{2}(P_hV_l - P_lV_l) > 0$	> 0	0
Cycle			$\frac{(P_h - P_l)(V_h - V_l)}{2}$

Assuming ideal gas law, the temperatures are:

$T_1 = \frac{P_hV_l}{nR}, T_2 = \frac{P_0V_h}{nR}, T_3 = \frac{P_lV_l}{nR}$

The heat in to the system:

$Q_{in} = Q_{12} + Q_{31} = \frac{3}{2}(P_0V_h - P_lV_l) + \frac{(P_h+P_0)(V_h-V_l)}{2} = 50\ kJ$

Therefore, $P_0 = 1\ E6\ Pa = P_l$. So $T_2 > T_1 > T_3$

The total work: $W = \frac{(P_h-P_l)(V_h-V_l)}{2} = 20E - 3 * \frac{1E6}{2} = 10\ kJ,$

Efficiency: $\eta = \frac{W}{Q_{in}} = \frac{10}{50} = 0.2$

In comparison: $1 - \frac{T_c}{T_h} = 1 - \frac{T_3}{T_2} = 1 - \frac{P_lV_l}{P_0V_h} = 0.8$

Thermo B4

According to Stefan-Boltzman law, power radiated by sun per unit solid angle is

$$P_{rad} = \sigma T_s^4 \frac{4\pi R_s^2}{4\pi} \quad \text{where } R_s \text{ is the sun's radius}$$

power absorbed by earth is

$$P_{abs} = P_{rad} \frac{\pi R_e^2}{d^2} \quad \text{where } R_e \text{ is the earth's radius}$$

$$P_{abs} = \sigma T_s^4 \frac{\pi R_e^2 R_s^2}{d^2}$$

power radiated by earth

$$P_{rad}' = \sigma T_e^4 \cdot 4\pi R_e^2 \quad \text{where } T_e \text{ is earth's temperature}$$

according to the blackbody radiation law

$$P_{abs} = P_{rad}'$$

$$\sigma T_s^4 \frac{\pi R_e^2 R_s^2}{d^2} = \sigma T_e^4 \, 4\pi R_e^2$$

$$T_e^4 = \frac{R_s^2}{4d^2} T_s^4$$

$$T_e = T_s \sqrt{\frac{R_s}{2d}} = 6\times10^3 \sqrt{\frac{6.96\times10^5}{3\times10^8}} = 288\,K$$

$$= 15°C$$