UNL - Department of Physics and Astronomy

Preliminary Examination - Day 1
Tuesday, May 28, 2024

This test covers the topics of *Thermodynamics and Statistical Mechanics* (Topic 1) and *Classical Mechanics* (Topic 2). Each topic has 4 “A” questions and 4 “B” questions. Work two problems from each group. Thus, you will work on a total of 8 questions today, 4 from each topic.

Note: If you do more than two problems in a group, only the first two (in the order they appear in this handout) will be graded. For instance, if you do problems A1, A3, and A4, only A1 and A3 will be graded.

WRITE YOUR ANSWERS ON ONE SIDE OF THE PAPER ONLY

Thermodynamics and Statistical Mechanics Group A

Answer only two Group A questions

A1. For a vacuum tube of pressure 2×10^{-3} Pa and temperature of 27 °C, calculate:

(a) number of gas particles per m^3;

(b) volume occupied per particle;

(c) mean free path of the particle (assuming the particle has a radius of 155 pm).

Hint: You need to calculate the scattering cross-section and scattering volume. There is a factor of $\sqrt{2}$ when considering the relative motion between particles.

A2. Find the thermal expansion coefficient $\alpha = (\partial V/\partial T)_P/V$ and isothermal compressibility $K_T = -(\partial V/\partial P)_T/V$ for ideal gas.

A3. A material's density is ρ_s and ρ_l for solid and liquid phase, respectively. Given that the latent heat for the solid-liquid transition is L/kg at pressure P, find the change of internal energy during the transition per kg.

A4. One mole of a monatomic perfect gas initially at temperature T_0 expands from volume V_0 to $2V_0$, (a) at constant temperature, (b) at constant pressure. Calculate the work of expansion and the heat absorbed by the gas in each case.

Thermodynamics and Statistical Mechanics Group B

Answer only two Group B questions

B1. A rigid adiabatic container is divided into two parts containing n_1 and n_2 mole of ideal gas, respectively, by a movable and thermally conducting wall. Their pressure and volume are P_1, V_1 for part 1 and P_2, V_2 for part 2, respectively. The wall was initially held in place and then released allowing frictionless motion. Find the final pressure P and temperature T after the two gases reaches equilibrium. Assume the constant-volume specific heats of the two gases are the same.

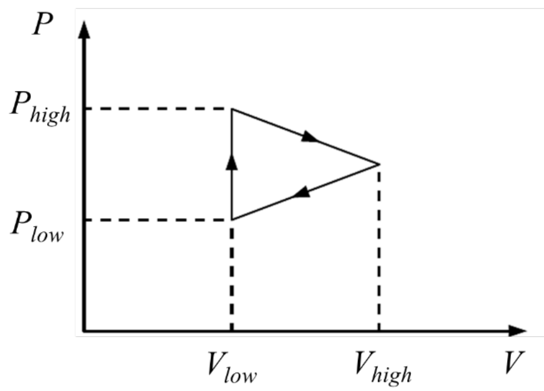
B2. In a throttling process, the Joule-Thompson coefficient is defined as $\mu = (\partial T/\partial P)_H$.

(a) Given the relation $(\partial H/\partial P)_T = V - T(\partial V/\partial T)_P$, show that $\mu = V(T\alpha - 1)/C_P$, where $\alpha = (\partial V/\partial T)_P/V$.

Hint: use the cyclic chain rule.

(b) Show that for ideal gas, $\mu = 0$.

B3. The cycle of a hypothetical engine is illustrated below. Let $P_{low} = 1\times10^6$ Pa, $P_{high} = 2\times10^6$ Pa, $V_{low} = 5\times10^{-3}$ m^3, and $V_{high} = 25\times10^{-3}$ m^3. What is the efficiency of the engine, if the energy absorbed by heating the engine is 5×10^4 J? Calculate the highest and lowest temperature (T_h and T_l) in the cycle and calculate $1 - \frac{T_l}{T_h}$, assuming that the pressure at V_{high} is $(P_{low} + P_{high})/2$.



B4. Calculate the average temperature of the planet earth treating both earth and sun as black bodies and using the following data: The sun's surface temperature is 6000 K, the sun's radius is 6.96×10^5 km, and the distance between sun and earth is 1.5×10^8 km.

Classical Mechanics Group A

Answer only two Group A questions

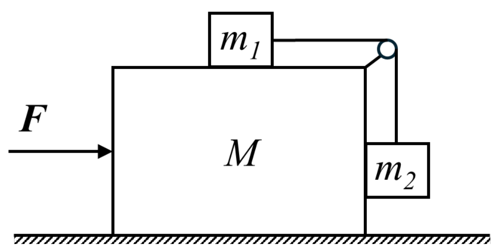
A1. A space ship of mass 5×10^3 kg is moving through a dust cloud causing the quadratic resistance $-cv^2$, where v is the speed of the ship and $c = 10^{-5}$ kg/m.

(a) How long will it take to reduce the ship's speed by a factor of 2, if the initial speed is 10^4 m/s?

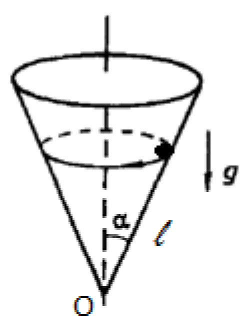
(b) Find the distance covered by the ship during this time.

A2. A small mass m rests at the edge of a horizontal rotating disk of radius R. The coefficient of static friction between the mass and the disk is μ. Due to a small perturbation the mass slides off the disk and lands on the floor h meters below. What was its horizontal distance of travel from the point that it left the disk? Neglect air resistance. Answer the same question if the mass is doubled while all the conditions in the problem remain the same. How would the horizontal distance change if this experiment is repeated on another planet?

A3. A body of mass M with plane surfaces is pushed with force F along the ground surface as depicted below. Two masses m_1 and m_2 are connected by a cord through a pulley and are able to slide along the top and side surfaces of the body, respectively, as shown. Assume all surfaces to be frictionless and the inertia of the pulley and the cord negligible. Find the horizontal force necessary to prevent any relative motion of m_1, m_2 and M.



A4. A particle of mass m is constrained to move on the frictionless inner surface of a cone of half-angle α. Assuming that the particle is on a circular orbit about the vertical axis, find distance l from the vertex O if velocity tangential to the circular orbit is v.



<u>Classical Mechanics Group B</u>

Answer only two Group B questions

B1. A particle of mass m is performing one-dimensional motion subject to the force

$$F(x) = -F_0 \sin(cx).$$

At some instant the particle's position is $x = 0$ and the velocity $v = v_0$.

(a) Find the potential energy as a function of x and sketch it.

(b) Find the velocity as a function of x.

(c) Find the condition on the initial velocity v_0 for which the motion is periodic. Find the turning points for the periodic motion.

(d) Suppose that v_0 is so small that the periodic motion can be treated as harmonic oscillations. Find the period of these oscillations and the oscillation amplitude.

B2. A cylinder of radius a is rolling down an inclined plane with angle θ without slipping.

(a) Choose a generalized coordinate and write the Lagrangian for this system.

(b) Write the Lagrange's equation of motion, solve it and obtain the cylinder's acceleration.

(c) Write the Hamiltonian of the system and the Hamilton's equations of motion. Show that their solution is equivalent to the solution obtained in part (b).

B3. An artillery shell is fired at an angle of elevation 60° with initial speed $v_0 = 100$ m/s. In the upper most part of its trajectory, the shell bursts into two equal fragments, one of which moves directly upward, relative to the ground with initial speed $v_0/2 = 50$ m/s.

(a) What is the direction and the speed of the other fragment immediately after the burst?

(b) Find the distance from the cannon to the point where the second fragment strikes the ground.

Note: to simplify calculations use $g = 10$ m/s^2.

B4. Two blocks of equal mass m are connected by a flexible, but non-stretchable, cord of mass m'. One block is placed on a smooth horizontal table, the other block hangs over the edge.

(a) Choose the generalized coordinate x as the displacement of either block from the initial position, and write the Lagrangian of the system;

(b) Obtain position x and velocity v as functions of time assuming that $x(0) = 0$, $v(0) = 0$.

Physical Constants

Speed of light .... $c = 2.998 \times 10^{8}$ m/s
Atmospheric pressure........... $101{,}325$ Pa
Electron mass .... $m_e = 9.109 \times 10^{-31}$ kg
Avogadro constant .... $N_{\mathrm{A}} = 6.022 \times 10^{23}$ mol^{-1}
Boltzmann constant.............. $k_{\mathrm{B}} = 1.381 \times 10^{-23}$ J / K = 8.617×10^{-5} eV/K
Gas constant .... $R = 8.314$ J / (mol·K)
Atomic mass unit .... $1\,\mathrm{u} = 1.66 \times 10^{-27}$ kg
Gravitational constant .... $G = 6.674 \times 10^{-11}$ m^3 / (kg·s^2) ; $g = 9.8$ m/s^2

Equations That May Be Helpful

TRIGONOMETRY

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$
$$\sin\alpha\sin\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \tfrac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \tfrac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

For small x:

$$\sin x \approx x - \tfrac{1}{6}x^3$$
$$\cos x \approx 1 - \tfrac{1}{2}x^2$$
$$\tan x \approx x + \tfrac{1}{3}x^3$$

THERMODYNAMICS

Heat capacity $C_V = N\dfrac{d\langle E\rangle}{dT}$.

Clausius' theorem: $\sum_{i=1}^{N} \frac{Q_i}{T_i} \leq 0$; becomes $\sum_{i=1}^{N} \frac{Q_i}{T_i} = 0$ for a reversible cyclic process of N steps.

$$\frac{dp}{dT} = \frac{\lambda}{T\Delta V}$$

Molar heat capacity of diatomic gas: $C_V = \frac{5}{2} R$.

For adiabatic processes in an ideal gas with constant heat capacity, $pV^{\gamma} = \text{const}$.

$$dU = TdS - pdV$$
$$dF = -SdT - pdV$$

$$H = U + pV$$
$$F = U - TS$$
$$G = F + pV$$
$$\Omega = F - \mu N$$

$$C_V = \left(\frac{\delta Q}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V$$
$$C_p = \left(\frac{\delta Q}{dT}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p$$
$$TdS = C_V dT + T\left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$
$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$$

Efficiency of a heat engine: $\eta = \frac{W}{|Q_{in}|} = 1 - \frac{|Q_{out}|}{|Q_{in}|}$

Carnot efficiency = $1 - T_c/T_h$.

The cyclic rule: $\left(\frac{\partial T}{\partial P}\right)_H \left(\frac{\partial P}{\partial H}\right)_T \left(\frac{\partial H}{\partial T}\right)_P = -1.$

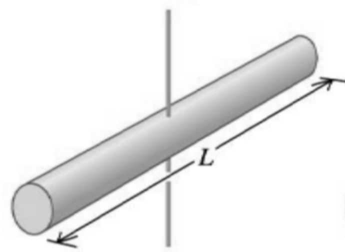
Stefan-Boltzmann's law:

$$P = \sigma T^4; \quad \sigma = 5.67 \times 10^{-8} W \cdot m^{-2} \cdot K^{-4}$$

TABLE 9.2 Moments of Inertia of Various Bodies

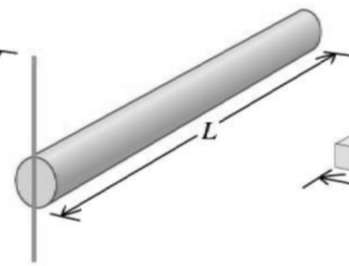
(a) Slender rod, axis through center

$$I = \frac{1}{12}ML^2$$



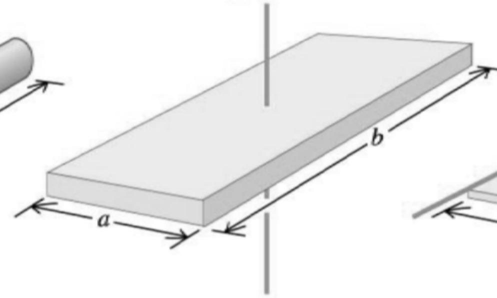
(b) Slender rod, axis through one end

$$I = \frac{1}{3}ML^2$$



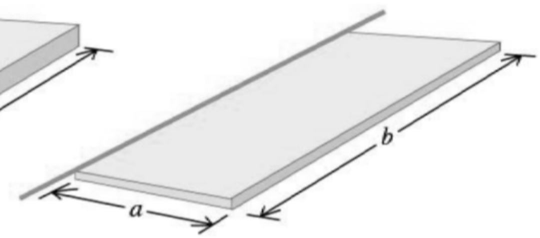
(c) Rectangular plate, axis through center

$$I = \frac{1}{12}M(a^2 + b^2)$$



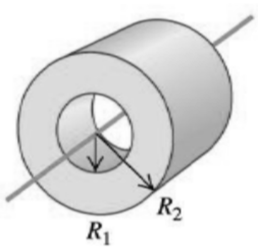
(d) Thin rectangular plate, axis along edge

$$I = \frac{1}{3}Ma^2$$



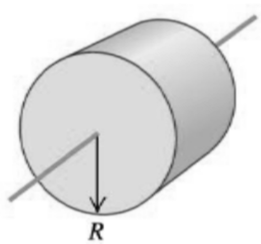
(e) Hollow cylinder

$$I = \frac{1}{2}M(R_1^2 + R_2^2)$$



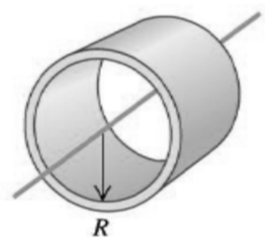
(f) Solid cylinder

$$I = \frac{1}{2}MR^2$$



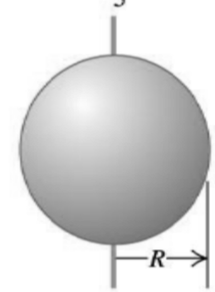
(g) Thin-walled hollow cylinder

$$I = MR^2$$



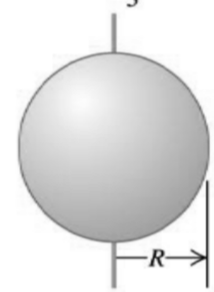
(h) Solid sphere

$$I = \frac{2}{5}MR^2$$



(i) Thin-walled hollow sphere

$$I = \frac{2}{3}MR^2$$



VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\,\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\,\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

CARTESIAN AND SPHERICAL UNIT VECTORS

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi)\hat{\mathbf{r}} + (\cos\theta\cos\phi)\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{y}} = (\sin\theta\sin\phi)\hat{\mathbf{r}} + (\cos\theta\sin\phi)\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

INTEGRALS

$\int_0^\infty \frac{1}{1+bx^2}\,dx = \frac{\pi}{2b^{1/2}}$	$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$
$\int_0^\infty x^n e^{-bx}\,dx = \frac{n!}{b^{n+1}}$	$\int_0^\infty x e^{-x^2}\,dx = \frac{1}{2a}$
$\int (x^2+b^2)^{-1/2}\,dx = \ln\left(x+\sqrt{x^2+b^2}\right)$	$\int_0^\infty x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2a^{3/2}}$
$\int (x^2+b^2)^{-1}\,dx = \frac{1}{b}\arctan\left(\frac{x}{b}\right)$	$\int_0^\infty x^3 e^{-x^2}\,dx = \frac{1}{2a^2}$
$\int (x^2+b^2)^{-3/2}\,dx = \frac{x}{b^2\sqrt{x^2+b^2}}$	$\int_0^\infty x^4 e^{-x^2}\,dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$
$\int (x^2+b^2)^{-2}\,dx = \frac{\frac{bx}{x^2+b^2}+\arctan\left(\frac{x}{b}\right)}{2b^3}$	$\int_0^\infty x^5 e^{-x^2}\,dx = \frac{1}{a^3}$
$\int \frac{x\,dx}{x^2+b^2} = \frac{1}{2}\ln\left(x^2+b^2\right)$	$\int_0^\infty x^6 e^{-x^2}\,dx = \frac{15\sqrt{\pi}}{16a^{7/2}}$
$\int \frac{dx}{x(x^2+b^2)} = \frac{1}{2b^2}\ln\left(\frac{x^2}{x^2+b^2}\right)$	
$\int \frac{dx}{a^2x^2-b^2} = \frac{1}{2ab}\ln\left(\frac{ax-b}{ax+b}\right) = -\frac{1}{ab}\operatorname{artanh}\left(\frac{ax}{b}\right)$	